Benha University | **Faculty of Computers & Informatics**

Final Exam

Date: 1/1/2017

Class: 1st Year Students **)** (تخلفات)

Time: 3 hours

Subject: Mathematics I

Examiner: Dr. El-Sayed Badr

Answer the following questions: (نموذج إجابة)

Q1:

1- Solve each inequality and sketch the graph of its solution.

a) $|x-3|<0.5$

Solution:

|x-3|<0.5

-0.5<x-3<0.5

2.5<x<3.5

b) $|2x-7|>3$

Solution:

2x-7 < -3 or 2x-7 > 3

2x < 4 or 2x > 10

x < 2 or x > 5

The solution are given by $(-\infty, 2)\cup(5,\infty)$.

2- Find $(f\circ g)(x)$ **where** $f(x)=x^2-3x$ **and** $g(x)=\sqrt{x+2}$.

Solution:

$$(f\circ g)(x)=f(f(x))=f(\sqrt{x+2})=(\sqrt{x+2})^2-3\sqrt{x+2}=x+2-3\sqrt{x+2}$$

3- Prove that $\lim_{x\to 3/2}\frac{4x^2-9}{2x-3}=6$.

Solution:

We can use the definition of limit to prove the above.

First we can simplify the limit $\lim_{x\to 3/2}\frac{4x^2-9}{2x-3}=6$ as $\lim_{x\to 3/2}\frac{(2x-3)(2x+3)}{2x-3}=6$ *then* $\lim_{x\to 3/2}2x+3=6$

$if\ 0<|x-3/2|<\delta \quad then\ |3x+2-6|<\varepsilon$

$|3x+2-6|<\varepsilon \quad then\ |3x-4|<\varepsilon \quad then\ |x-4/3|<\varepsilon/3$ then take $\delta=\varepsilon/3$.

Q2:

1- Write and explain the Sandwich Theorem using an example ?

Solution:

Suppose $f(x) \le h(x) \le g(x)$ for every x in an open interval containing a except at a.

If $\lim_{x \to a} f(x) = L = \lim_{x \to a} g(x), then \quad \lim_{x \to a} h(x) = L.$

For example : $\lim_{x \to 0} x^2 \sin \frac{1}{x^2}$.

2- Find $\lim_{x \to \infty} \frac{\sqrt{9x^2+2}}{4x+3}$ and $\lim_{x \to -\infty} \frac{\sqrt{9x^2+2}}{4x+3}$

Solution:

$$\lim_{x \to \infty} \frac{\sqrt{9x^2+2}}{4x+3} = \lim_{x \to \infty} \frac{\sqrt{x^2(9+2/x^2)}}{4x+3} = \lim_{x \to \infty} \frac{\sqrt{x^2}\sqrt{(9+2/x^2)}}{4x+3}$$

If x is positive, then $\sqrt{x^2} = x$, and dividing numerator and denominator of the last fraction by x gives us

$$\lim_{x \to \infty} \frac{\sqrt{9x^2+2}}{4x+3} = \lim_{x \to \infty} \frac{\sqrt{(9+2/x^2)}}{4+3/x}$$

$$= \frac{\sqrt{9+0}}{4+0} = \frac{3}{4}$$

If x is large negative, then $\sqrt{x^2}$ = -x , and dividing numerator and denominator of the last fraction by x gives us

$$\lim_{x \to -\infty} \frac{\sqrt{9x^2+2}}{4x+3} = \lim_{x \to -\infty} \frac{\sqrt{x^2(9+2/x^2)}}{4x+3} = \lim_{x \to -\infty} \frac{\sqrt{x^2}\sqrt{(9+2/x^2)}}{4x+3}$$

$$\lim_{x \to -\infty} \frac{\sqrt{9x^2+2}}{4x+3} = \lim_{x \to -\infty} \frac{(-x)\sqrt{(9+2/x^2)}}{4+3/x}$$

$$= \frac{-\sqrt{9+0}}{4+0} = \frac{-3}{4}$$

3- Study the continuity of the function $f(x) = \sqrt{9-x^2}$ on the closed interval [-3, 3].

Solution:

For any point c such that -3 < c < 3, then $\lim_{x \to c} \sqrt{9-x^2} = \sqrt{9-c^2} = f(c).$

Hence f is continuous at c . All that remains is to check the endpoints of the interval {-3,3] using one-side limits as follows:

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9-x^2} = \sqrt{9-9} = 0 = f(-3).$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9-x^2} = \sqrt{9-9} = 0 = f(3).$$

Thus, f is continuous from the right at -3 and from the left at 3. From all above, Thus the function f is continuous on [-3,3].

4- If $y = (x^3+1)(2x^2+8x-5)$, find $D_x(y)$.

Solution:

$$D_X\, y = (x^3+1)\, D_X\,(2x^2+8x-5) + (2x^2+8x-5) D_X\,(x^3+1)$$
$$= (x^3+1)(4x+8) + (2x^2+8x-5)(3x^2)$$
$$= (4x^4+8x^3+4x+8) + (6x^4+24x^3-15x^2)$$
$$= (10x^4+32x^3-15x^2+4x+8)$$

Q3:

1- Find $\frac{dy}{dx}$ if $y = \frac{3x^2-x+2}{4x^2+5}$

Solution:

$$\frac{dy}{dx} = \frac{(4x^2+5)D_x(3x^2-x+2)-(3x^2-x+2)D_x(4x^2+5)}{(4x^2+5)^2}$$
$$= \frac{(4x^2+5)(6x-1)-(3x^2-x+2)(8x)}{(4x^2+5)^2}$$
$$= \frac{(24x^3-4x^2+30x-5)-(24x^3-8x^2+16x)}{(4x^2+5)^2}$$
$$= \frac{4x^2+14x-5}{(4x^2+5)^2}$$

2- Prove that $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

3- Find y' if $4xy^3 - x^2y + x^3 - 5x + 6 = 0$

Solution:

$$D_x(4xy^3) - D_x(x^2y) + D_x(x^3) - D_x(5x) + D_x(6) = 0$$
$$D_x(4xy^3) = 12xy^2y' + 4y^3$$
$$D_x(x^2y) = x^2y' + y(2x)$$
$$D_x(x^3) = 3x^2$$
$$D_x(5x) = 5$$
$$D_x(6) = 0$$

So we get :

$$(12xy^2y' + 4y^3) - (x^2y' + y(2x)) + 3x^2 - 5 = 0$$
$$(12xy^2 - x^2)y' = 5 - 3x^2 + 2xy - 4y^3$$
$$y' = \frac{5-3x^2+2xy-4y^3}{(12xy^2-x^2)}$$ provided $12xy^2 - x^2 \neq 0$

4- Evaluate :

a) $\int \frac{1}{\cos(x)\cot(x)}\,dx$

Solution:

$$\int \frac{1}{\cos(x)\cot(x)}\,dx = \int \sec x \tan x \, dx = \sec x + c$$

b) $\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx$

Solution:

Let

$u = x^3 - 3x + 1 \qquad then \quad du = 3(x^2-1)dx$

so

$$\int \frac{x^2-1}{(x^3-3x+1)^6}\,dx = \frac{1}{3}\int \frac{3(x^2-1)}{(x^3-3x+1)^6}\,dx = \frac{1}{3}\int \frac{1}{(u)^6}\,dx$$

$$= \frac{1}{3}\int u^{-6}\,dx = \frac{1}{3}\left(\frac{u^{-5}}{-5}\right) = \frac{-1}{15}\left(\frac{1}{u^5}\right) + C$$

$$= \frac{-1}{(15}\frac{1}{x^3-3x+1)^5} + C$$

c) $\int_0^{\pi/4} (1+\sin 2x)^3 dx$

Solution:

$$\int_0^{\pi/4} (1+\sin 2x)^3 dx = \frac{1}{2}\int_1^2 u^3 du = \frac{1}{2}\Big[(u^4/4)\Big]_1^2$$

$$= \frac{1}{8}(16-1) = 15/8$$

d) $\int_{-1}^{2} (x^3+1)^2 dx$

Solution:

$$\int_{-1}^{2} (x^3+1)^2 dx = \int_{-1}^{2} (x^6+2x^3+1)dx = \left[\left(\frac{x^7}{7} + 2\frac{x^4}{4} + x\right)\right]_{-1}^{2}$$

$$= \frac{405}{14}$$

Q4:

1- Solve the differential equation $f'' = 5\cos x + 2\sin x$ subject to the initial conditions $f(0) = 3$ *and* $f'(0) = 4$.

Solution:

$f'' = 5\cos x + 2\sin x$

$\int f''dx = \int (5\cos x + 2\sin x)dx$

$f'(x) = 5\sin x - 2\cos x + c$

Letting x = 0 and using the initial condition $f'(0) = 4$ given us

$f'(0) = 5\sin 0 - 2\cos 0 + c$

$4 = 0 - 2*1 + c$

$c = 6$

Hence

$f'(x)=5\sin x-2\cos x+6$

So

$f'(x)=5\sin x-2\cos x+6$

$\int f'(x)dx=\int(5\sin x-2\cos x+6)dx$

$f(x)=-5\cos x-2\sin x+6x+D$

Letting x = 0 and using the initial condition $f(0)=3$ given us

$f(0)=-5\cos 0-2\sin 0+D$

$3=-5-0+0+D$

$D=8$

So $f(x)=-5\cos x-2\sin x+6x+8$

2- Evaluate :

a) $\int\frac{e^{2x}}{\sqrt{1-e^{4x}}}dx$

Solution:

Let u = e^{2x} , du = 2 e^{2x} dx

$$\int\frac{e^{2x}}{\sqrt{1-e^{4x}}}dx=\frac{1}{2}\int\frac{1}{\sqrt{1-u^2}}dx=\frac{1}{2}\int\frac{1}{\sqrt{1-u^2}}du=\frac{1}{2}\sin^{-1}u+C$$

$$\frac{1}{2}\sin^{-1}u+C=\frac{1}{2}\sin^{-1}(e^{2x})+C$$

b) $\int\frac{e^{x}}{16-e^{2x}}dx$

Solution:

Let u = e^{x} , du = e^{x} dx

$$\int\frac{1}{16-u^2}dx=\int\frac{1}{4^2-u^2}dx=\frac{1}{4}\tanh(\frac{u}{4})+C$$

$$\frac{1}{4}\tanh(\frac{u}{4})+C=\frac{1}{4}\tanh(\frac{e^x}{4})+C$$

3- If $f(x)=\cosh(x^2+1)$ find $f'(x)$.

Solution:

$f'(x)=\cosh(x^2+1)D_x(x^2+1)$

$=2x\sinh(x^2+1)$

4- If $y=\sinh^{-1}(\tan x)$ find dy/dx

Solution:

$$dy/dx=\frac{1}{\sqrt{\tan^2 x+1}}D_x(\tan x)=\frac{\sec^2 x}{\sec x}=|\sec x|$$

Good Luck